Recall: Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \sin^2 x = 1 - \cos^2 x \quad \Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x \quad \Rightarrow \tan^2 x = \sec^2 x - 1$$

Recall: Double Angle (Reduction) Formulas

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = 2\cos^2 x - 1 \quad \Rightarrow \cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

$$\cos 2x = 1 - 2\sin^2 x \quad \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Guidelines for Trigonometric Substitutions –

look at format under the radical:

Given:	**Use:**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Sometimes, the above formats appear without the radical…

ex. $\int \frac{x^2\,dx}{1+x^2}$ $a=1$

" a^2+x^2 "

$x=\tan\theta$ $x^2=\tan^2\theta$

$dx=\sec^2\theta\,d\theta$

$= \int \frac{\tan^2\theta \sec^2\theta\, d\theta}{1+\tan^2\theta}$

from p.1: $1+\tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$= \int \frac{\tan^2\theta\, \cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\, d\theta$

random thought: $\int 1\, d\theta = \int d\theta$

$= \int \tan^2\theta\, d\theta$

$= \int (\sec^2\theta - 1)\, d\theta$

derivative

$= \int \sec^2\theta\, d\theta - \int 1\, d\theta$

$= \tan\theta - \theta + C$

$x=\tan\theta \Rightarrow \arctan x = \theta$

$= \boxed{x - \arctan x + C}$

ex. $\int \frac{dx}{x^2\sqrt{25+x^2}}$ "a^2+x^2"

$a^2=25 \Rightarrow a=5$

$x = 5\tan\theta$ $\quad x^2 = 25\tan^2\theta$

$dx = 5\sec^2\theta\, d\theta$

$= \frac{5}{25}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{25+25\tan^2\theta}}$ → Do: simplify until radical cancels

$\sqrt{25+x^2} = \sqrt{25+25\tan^2\theta}$

$= \sqrt{25(1+\tan^2\theta)}$

$= \sqrt{25}\sqrt{1+\tan^2\theta}$ $\quad 1+\tan^2\theta = \sec^2\theta$

$= 5\sqrt{\sec^2\theta}$

$= 5\sec\theta$

$= \frac{1}{5}\cdot\frac{1}{5}\int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta\,\cancel{\sec\theta}}\, d\theta$

$= \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$

$= \frac{1}{25}\int \sec\theta\cdot\frac{1}{\tan\theta}\cdot\frac{1}{\tan\theta}\, d\theta$ $\quad \cot\theta$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$= \frac{1}{25}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\, d\theta$

$= \frac{1}{25}\int \frac{\cos\theta\, d\theta}{\sin^2\theta}$ $\quad u = \sin\theta \quad du = \cos\theta\, d\theta$

$= \frac{1}{25}\int \frac{1}{u^2}\, du$

$= \frac{1}{25}\int u^{-2}\, du$

$= \frac{1}{25}\frac{u^{-1}}{-1} + C = -\frac{1}{25}\cdot\frac{1}{u} + C = -\frac{1}{25}\cdot\frac{1}{\sin\theta} + C$

$= -\frac{1}{25}\cdot\boxed{\frac{1}{\frac{x}{\sqrt{x^2+25}}}} + C$ KCF

$= -\frac{\sqrt{x^2+25}}{25x} + C$

$x = 5\tan\theta$

$\frac{O}{A}\quad \frac{x}{5} = \tan\theta$

$x^2+5^2 = H^2$

$H = \sqrt{x^2+25}$

$\sin\theta = \frac{O}{H}$

$= \frac{x}{\sqrt{x^2+25}}$

ex. $\displaystyle\int \frac{x^3\,dx}{\left(1+x^2\right)^{5/2}}$ $a=1$

$x = \tan\theta$ $x^2 = \tan^2\theta$ $x^3 = \tan^3\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\tan^3\theta \sec^2\theta\, d\theta}{(1+\tan^2\theta)^{5/2}}$$

$1+\tan^2\theta = \sec^2\theta$

$$= \int \frac{\tan^3\theta \sec^2\theta\, d\theta}{(\sec^2\theta)^{5/2}}$$

$(x^2)^{5/2} = x^{\cancel{2}\cdot\frac{5}{\cancel{2}}} = x^5$

$$= \int \frac{\tan^3\theta\, \cancel{\sec^2\theta}}{\sec^{\cancel{5}\,3}\theta}\, d\theta$$

$$= \int \frac{\tan^3\theta}{\sec^3\theta}\, d\theta$$

Do write ITO $\sin\theta, \cos\theta$

$$= \int \frac{\sin^3\theta}{\cancel{\cos^3\theta}} \cdot \cancel{\cos^3\theta}\, d\theta$$

$$= \int \sin^3\theta\, d\theta = \int \underbrace{\sin^2\theta}_{\text{write ITO } \cos\theta}\, \underbrace{\sin\theta\, d\theta}_{du}$$

$u = \cos\theta$

$-du = \sin\theta\, d\theta$

$$= \int (1-\cos^2\theta)\sin\theta\, d\theta$$

$$= -\int (1-u^2)\, du$$

$$= \int (-1+u^2)\, du$$

$\int 1\, du = u$

$$= -u + \frac{u^3}{3} + C$$

$$= \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$x = \tan\theta \Rightarrow \tan\theta = \frac{x}{1} \frac{O}{A}$

Do: write $\cos\theta$ ITO x

$H = \sqrt{1+x^2}$

$\therefore \cos\theta = \frac{A}{H} = \frac{1}{\sqrt{1+x^2}}$

$$= \frac{1}{3}\cdot\left(\frac{1}{\sqrt{1+x^2}}\right)^3 - \frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{3}\cdot\frac{1}{(1+x^2)^{3/2}} - \frac{3}{3}\cdot\frac{1}{(1+x^2)^{1/2}}\cdot\frac{(1+x^2)^3}{(1+x^2)^3} + C$$

$$= \boxed{\frac{1-3(1+x^2)^3}{3(1+x^2)^{3/2}} + C}$$